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Hamiltonian symplectic embedding of the massive noncommutative $U(1)$ theory

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Abstract
We show that the massive noncommutative $U(1)$ theory is embedded in a gauge theory using an alternative systematic way [1], which is based on the symplectic framework. The embedded Hamiltonian density is obtained after a finite number of steps in the iterative symplectic process as opposed to the result proposed using the BFFT formalism [2]. This alternative formalism of embedding shows how to get a set of dynamically equivalent embedded Hamiltonian densities.

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1. Introduction

The embedding mechanism, first suggested by Faddeev and Shatashivilli [3], has been a successful constraint conversion procedure over the last decades. The main concept behind this procedure is based on the enlargement of the phase space with the introduction of new variables, called Wess–Zumino (WZ) variables, which changes the second class nature of the constraint to first class. This procedure has been developed in different contexts [1, 7–9] in order to avoid some problems that affect the quantization process of some theories, such as chiral theory, where the anomaly obstructs the quantization mechanism, and nonlinear models, where the operator ordering ambiguities arise. It is important to comment here that the proposed embedding procedure, whether it is applied to commutative or noncommutative theories, is to unveil the origin of the ambiguities of all embedding approaches.

The great deal of interest in noncommutative (NC) field theories started when it was noted that noncommutative spaces naturally arise in string theory with a constant background magnetic field in the presence of D-branes. It is important to mention here that this noncommutativity in the context of string theory with a constant background magnetic field in the presence of D-branes is eliminated constructing a mechanical system which reproduces classical dynamics of the string [10]. Besides their origin in strings theories and branes, NC field theories have been extensively studied in many branches of physics [2, 4–6, 11, 12].

In order to obtain the noncommutative version of a field theory one replaces the usual product of fields in the action by the Moyal product, defined as

$$\phi_1(x) \star \phi_2(x) = \exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y\right)\phi_1(x)\phi_2(y)\Big|_{x=y} \tag{1}$$

where $\theta^{\mu\nu}$ is a real and antisymmetric constant matrix. As a consequence, NC theories are highly nonlocal. We also note that the Moyal product of two fields in the action is the same as the usual product (see the appendix), provided we discard boundary terms. Thus, the noncommutativity affects just the vertices.

Recently, the embedded version of the massive noncommutative $U(1)$ theory was obtained through the BFFT constraint conversion scheme [2]. In this work, the authors showed how to obtain a set of second class constraints and Hamiltonian which form an involutive system of dynamical quantities. However, both the constraints and Hamiltonian were expressed as a series of Moyal commutators among the variables belonging to the WZ extended phase space. Our goal in the present work is to propose an embedded version for the massive noncommutative $U(1)$ theory where the embedded Hamiltonian density is not expressed as an expansion on the WZ variables but as a finite sum. To this end, we will use the symplectic embedding formalism (see section 2) [1].

Our paper is organized as follows. In section 2, we present an overview of the symplectic embedding formalism. In section 3, we analyse the symplectic quantization of the noncommutative massive $U(1)$ theory and compute the Dirac brackets among the phase space. In section 4, we investigate the embedded version for the noncommutative massive $U(1)$ from the symplectic embedding point of view. We note that after a finite number of steps of the iterative symplectic embedding process, we obtain an embedded Hamiltonian density. In consequence, this Hamiltonian density has a finite number of WZ terms as opposed to [2]. In section 5, we present some concluding remarks. In the appendix, we list some properties of the Moyal product that we use in this paper.

2. General formalism

In this section, we describe an alternative embedding technique that changes the second class nature of a constrained system to first class. This technique follows the Faddeev and Shatashivilli formalism [3] and is based on a contemporary framework that handles constrained models, namely, the symplectic formalism [13, 15].

In order to systematize the symplectic embedding formalism, we consider a general noninvariant mechanical model whose dynamics is governed by a Lagrangian $\mathcal{L}(a_i, \dot{a}_i, t)$ (with $i = 1, 2, \ldots, N$), where a_i and $\dot{a}_i$ are the space and velocity variables, respectively. Note that this model does not result in loss of generality or physical content. Following the symplectic method, the zeroth-iterative first-order Lagrangian 1-form is written as

$$\mathcal{L}^{(0)}\,\mathrm{d}t = A_\theta^{(0)}\,\mathrm{d}\xi^{(0)\theta} - V^{(0)}(\xi)\,\mathrm{d}t, \tag{2}$$

where the symplectic variables are

$$\xi^{(0)\alpha} = \begin{cases} a_i, & \text{with } \alpha = 1, 2, \ldots, N, \\ p_i, & \text{with } \alpha = N+1, N+2, \ldots, 2N, \end{cases} \tag{3}$$

$A^{(0)}_\alpha$ are the canonical momenta and $V^{(0)}$ is the symplectic potential. The symplectic tensor is given by

$$f^{(0)}_{\alpha\beta} = \frac{\partial A^{(0)}_\beta}{\partial \xi^{(0)\alpha}} - \frac{\partial A^{(0)}_\alpha}{\partial \xi^{(0)\beta}}. \tag{4}$$

When the 2-form $f \equiv \frac{1}{2} f_{\theta\beta}\, \mathrm{d}\xi^\theta \wedge \mathrm{d}\xi^\beta$ is singular, the symplectic matrix (4) has a zero mode ($\nu^{(0)}$) that generates a new constraint when contracted with the gradient of the symplectic potential,

$$\Omega^{(0)} = \nu^{(0)\alpha} \frac{\partial V^{(0)}}{\partial \xi^{(0)\alpha}}. \tag{5}$$

This constraint is introduced into the zeroth-iterative Lagrangian 1-form, equation (2), through a Lagrange multiplier η, generating the next one

$$\begin{aligned} \mathcal{L}^{(1)}\, \mathrm{d}t &= A^{(0)}_\theta\, \mathrm{d}\xi^{(0)\theta} + \mathrm{d}\eta\, \Omega^{(0)} - V^{(0)}(\xi)\, \mathrm{d}t, \\ &= A^{(1)}_\gamma\, \mathrm{d}\xi^{(1)\gamma} - V^{(1)}(\xi)\, \mathrm{d}t, \end{aligned} \tag{6}$$

with $\gamma = 1, 2, \ldots, (2N+1)$ and

$$V^{(1)} = V^{(0)}|_{\Omega^{(0)}=0}, \qquad \xi^{(1)_\gamma} = (\xi^{(0)\alpha}, \eta), \qquad A^{(1)}_\gamma = \left(A^{(0)}_\alpha, \Omega^{(0)}\right). \tag{7}$$

As a consequence, the first-iterative symplectic tensor is computed as

$$f^{(1)}_{\gamma\beta} = \frac{\partial A^{(1)}_\beta}{\partial \xi^{(1)\gamma}} - \frac{\partial A^{(1)}_\gamma}{\partial \xi^{(1)\beta}}. \tag{8}$$

If this tensor is nonsingular, the iterative process stops and the Dirac brackets among the phase space variables are obtained by the inverse matrix $\left(f^{(1)}_{\gamma\beta}\right)^{-1}$ and, consequently, the Hamilton equation of motion can be computed and solved as well [14]. It is well known that a physical system can be described in terms of a symplectic manifold M, at least classically. From a physical point of view, M is the phase space of the system while a nondegenerate closed 2-form f can be identified as the Poisson bracket. The dynamics of the system is determined by just specifying a real-valued function (Hamiltonian) H on phase space, i.e., one of these real-valued functions solves the Hamilton equation, namely,

$$\iota(X) f = \mathrm{d}H, \tag{9}$$

and the classical dynamical trajectories of the system in phase space are obtained. It is important to mention here that if f is nondegenerate, equation (9) has a unique solution. The nondegeneracy of f means that the linear map $\flat : TM \to T^*M$ defined by $\flat(X) := \flat(X) f$ is an isomorphism; due to this equation (9) is solved uniquely for any Hamiltonian ($X = \flat^{-1}(\mathrm{d}H)$). In contrast, the tensor has a zero mode and a new constraint arises, indicating that the iterative process goes on until the symplectic matrix becomes nonsingular or singular. If this matrix is nonsingular, the Dirac brackets will be determined. In [14], the authors consider in detail the case when f is degenerate, which usually arises when constraints are presented on the system. In such a case, (M, f) is called a presymplectic manifold. As a consequence, the Hamilton equation, equation (9), may not possess solutions, or possess nonunique solutions. In contrast, if this matrix is singular and the respective zero mode does not generate a new constraint, the system has a symmetry.

The systematization of the symplectic embedding formalism begins by assuming that the gauge-invariant version of the general Lagrangian ($\tilde{\mathcal{L}}(a_i, \dot{a}_i, t)$) is given by

$$\tilde{\mathcal{L}}(a_i, \dot{a}_i, \varphi_p, t) = \mathcal{L}(a_i, \dot{a}_i, t) + \mathcal{L}_{\mathrm{WZ}}(a_i, \dot{a}_i, \varphi_p), \qquad p = 1, 2, \tag{10}$$

where $\varphi_p = (\theta, \dot{\theta})$ and the extra term ($\mathcal{L}_{\rm WZ}$) depends on the original $(a_i, \dot{a}_i)$ and WZ (φ_p) configuration variables. Indeed, this WZ Lagrangian can be expressed as an expansion in terms of the WZ variable (φ_p) such that

$$\mathcal{L}_{\rm WZ}(a_i, \dot{a}_i, \varphi_p) = \sum_{n=1}^{\infty} \upsilon^{(n)}(a_i, \dot{a}_i, \varphi_p), \qquad \text{with} \quad \upsilon^{(n)}(\varphi_p) \sim \varphi_p^n, \tag{11}$$

which satisfies the following boundary condition:

$$\mathcal{L}_{\rm WZ}(\varphi_p = 0) = 0. \tag{12}$$

The reduction of the Lagrangian, equation (10), into its first-order form precedes the beginning of the conversion process, thus

$$\tilde{\mathcal{L}}^{(0)}\,{\rm d}t = A^{(0)}_{\tilde{\alpha}}\,{\rm d}\tilde{\xi}^{(0)\tilde{\alpha}} + \pi_\theta\,{\rm d}\theta - \tilde{V}^{(0)}\,{\rm d}t, \tag{13}$$

where π_θ is the canonical momentum conjugated to the WZ variable, that is,

$$\pi_\theta = \frac{\partial \mathcal{L}_{\rm WZ}}{\partial \dot{\theta}} = \sum_{n=1}^{\infty} \frac{\partial \upsilon^{(n)}(a_i, \dot{a}_i, \varphi_p)}{\partial \dot{\theta}}. \tag{14}$$

The expanded symplectic variables are $\tilde{\xi}^{(0)\tilde{\alpha}} \equiv (a_i, p_i, \varphi_p)$ and the new symplectic potential becomes

$$\tilde{V}^{(0)} = V^{(0)} + G(a_i, p_i, \lambda_p), \qquad p = 1, 2, \tag{15}$$

where $\lambda_p = (\theta, \pi_\theta)$. The arbitrary function $G(a_i, p_i, \lambda_p)$ is expressed as an expansion in terms of the WZ fields, namely,

$$G(a_i, p_i, \lambda_p) = \sum_{n=0}^{\infty} \mathcal{G}^{(n)}(a_i, p_i, \lambda_p), \tag{16}$$

with

$$\mathcal{G}^{(n)}(a_i, p_i, \lambda_p) \sim \lambda_p^n. \tag{17}$$

In this context, the zeroth canonical momenta are given by

$$\tilde{A}^{(0)}_{\tilde{\alpha}} = \begin{cases} A^{(0)}_{\alpha}, & \text{with} \quad \tilde{\alpha} = 1, 2, \ldots, N, \\ \pi_\theta, & \text{with} \quad \tilde{\alpha} = N+1, \\ 0, & \text{with} \quad \tilde{\alpha} = N+2. \end{cases} \tag{18}$$

The corresponding symplectic tensor, obtained from the following general relation:

$$\tilde{f}^{(0)}_{\tilde{\alpha}\tilde{\beta}} = \frac{\partial \tilde{A}^{(0)}_{\tilde{\beta}}}{\partial \tilde{\xi}^{(0)\tilde{\alpha}}} - \frac{\partial \tilde{A}^{(0)}_{\tilde{\alpha}}}{\partial \tilde{\xi}^{(0)\tilde{\beta}}}, \tag{19}$$

is

$$\tilde{f}^{(0)}_{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} f^{(0)}_{\alpha\beta} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{20}$$

which should be a singular matrix.

The implementation of the symplectic embedding scheme consists in computing the arbitrary function $(G(a_i, p_i, \lambda_p))$. To this end, the correction terms in order of λ_p, within $\mathcal{G}^{(n)}(a_i, p_i, \lambda_p)$, must be computed as well. If the symplectic matrix (equation (20)) is singular, it has a zero-mode $\tilde{\varrho}$ and, consequently, we have

$$\tilde{\varrho}^{(0)\tilde{\alpha}} \tilde{f}^{(0)}_{\tilde{\alpha}\tilde{\beta}} = 0, \tag{21}$$

where we assume that this zero mode is

$$\tilde{\varrho}^{(0)} = (\gamma^\alpha \;\; 0 \;\; 0), \tag{22}$$

where γ^α is a generic line matrix. Using the relation given in equation (21) together with equations (20) and (22), we get

$$\gamma^\alpha f^{(0)}_{\alpha\beta} = 0. \tag{23}$$

In agreement with the symplectic formalism, we get a zero mode, which must be contracted with the gradient of the symplectic potential. Consequently, we obtain a constraint, which is given by

$$\Omega = \gamma^\alpha \left[\frac{\partial V^{(0)}}{\partial \xi^{(0)\alpha}} + \frac{\partial G(a_i, p_i, \lambda_p)}{\partial \xi^{(0)\alpha}} \right]. \tag{24}$$

Due to this, the first-order Lagrangian is rewritten as

$$\tilde{\mathcal{L}}^{(1)} = A^{(0)}_{\tilde{\alpha}} \dot{\xi}^{(0)\tilde{\alpha}} + \pi_\theta \dot{\theta} + \Omega \dot{\eta} - \tilde{V}^{(1)}, \tag{25}$$

where $\tilde{V}^{(1)} = V^{(0)}$. Note that the symplectic variables are now $\tilde{\xi}^{(1)\tilde{\alpha}} \equiv (a_i, p_i, \eta, \lambda_p)$ (with $\tilde{\alpha} = 1, 2, \ldots, N+3$) and the corresponding symplectic matrix becomes

$$\tilde{f}^{(1)}_{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} f^{(0)}_{\alpha\beta} & f_{\alpha\eta} & 0 & 0 \\ f_{\eta\beta} & 0 & f_{\eta\theta} & f_{\eta\pi_\theta} \\ 0 & f_{\theta\eta} & 0 & -1 \\ 0 & f_{\pi_\theta\eta} & 1 & 0 \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned} f_{\eta\theta} &= -\frac{\partial}{\partial\theta} \left[\gamma^\alpha \left(\frac{\partial V^{(0)}}{\partial \xi^{(0)\alpha}} + \frac{\partial G(a_i, p_i, \lambda_p)}{\partial \xi^{(0)\alpha}} \right) \right], \\ f_{\eta\pi_\theta} &= -\frac{\partial}{\partial\pi_\theta} \left[\gamma^\alpha \left(\frac{\partial V^{(0)}}{\partial \xi^{(0)\alpha}} + \frac{\partial G(a_i, p_i, \lambda_p)}{\partial \xi^{(0)\alpha}} \right) \right], \\ f_{\alpha\eta} &= \frac{\partial \Omega}{\partial \xi^{(0)\alpha}} = \frac{\partial}{\partial \xi^{(0)\alpha}} \left[\gamma^\alpha \left(\frac{\partial V^{(0)}}{\partial \xi^{(0)\alpha}} + \frac{\partial G(a_i, p_i, \lambda_p)}{\partial \xi^{(0)\alpha}} \right) \right]. \end{aligned} \tag{27}$$

Since our goal is to unveil a WZ symmetry, this symplectic tensor must be singular and, consequently, it has a zero mode, namely,

$$\tilde{\nu}^{(1)}_{(\nu)(a)} = \left(\mu^\alpha_{(\nu)} \quad 1 \quad a \quad b \right), \tag{28}$$

which satisfies the relation

$$\tilde{\nu}^{(1)\tilde{\alpha}}_{(\nu)(a)} \tilde{f}^{(1)}_{\tilde{\alpha}\tilde{\beta}} = 0. \tag{29}$$

Note that the parameters (a, b) can be 0 or 1 and ν indicates the number of choices for $\tilde{\nu}^{(1)\tilde{\alpha}}$. It is important to note that ν is not a fixed parameter. As a consequence, there are two independent sets of zero modes given by

$$\tilde{\nu}^{(1)}_{(\nu)(0)} = \left(\mu^\alpha_{(\nu)} \quad 1 \quad 0 \quad 1 \right), \qquad \tilde{\nu}^{(1)}_{(\nu)(1)} = \left(\mu^\alpha_{(\nu)} \quad 1 \quad 1 \quad 0 \right). \tag{30}$$

Note that the matrix elements $\mu^\alpha_{(\nu)}$ present some arbitrariness which can be fixed in order to disclose a desired WZ gauge symmetry. In addition, in our formalism the zero-mode $\tilde{\nu}^{(1)\tilde{\alpha}}_{(\nu)(a)}$ is the gauge symmetry generator, which allows us to display the symmetry from the geometrical point of view. At this point, we focus attention upon the fact that this is an important characteristic since it opens up the possibility of disclosing the desired hidden gauge symmetry from the noninvariant model. Different choices of the zero mode generate different gauge-invariant versions of the second class system; however, these gauge-invariant descriptions are dynamically equivalent, i.e., there is the possibility of relating this set of independent zero modes, equation (30), through canonical transformation $\left(\tilde{\bar{\nu}}^{(\prime,1)}_{(\nu)(a)} = T \cdot \tilde{\bar{\nu}}^{(1)}_{(\nu)(a)}\right)$, where a bar means a transpose matrix. For example,

$$\begin{pmatrix} \mu^{\alpha}_{(\nu)} \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mu^{\alpha}_{(\nu)} \\ 1 \\ 1 \\ 0 \end{pmatrix}. \tag{31}$$

It is important to mention here that, in the context of the BFFT formalism, the matrix X is arbitrary and, in consequence, different choices for the degenerate matrix X lead to different gauge-invariant versions of the second class model [16]. Now, it becomes clear that the arbitrariness present in the BFFT and iterative constraint conversion methods has its origin in the choice of the zero mode.

From relation (29), together with equations (26) and (28), some differential equations involving $G(a_i, p_i, \lambda_p)$ are obtained, namely,

$$\begin{aligned} 0 &= \mu^{\alpha}_{(\nu)} f^{(0)}_{\alpha\beta} + f_{\eta\beta}, \\ 0 &= \mu^{\alpha}_{(\nu)} f^{(0)}_{\alpha\eta} + a f_{\theta\eta} + b f_{\pi_\theta\eta}, \\ 0 &= f^{(0)}_{\eta\theta} + b, \\ 0 &= f^{(0)}_{\eta\pi_\theta} - a. \end{aligned} \tag{32}$$

By solving the above relations, some correction terms, within $\sum_{m=0}^{\infty} \mathcal{G}^{(m)}(a_i, p_i, \lambda_p)$, can be determined, also including the boundary conditions $(\mathcal{G}^{(0)}(a_i, p_i, \lambda_p = 0))$.

In order to compute the remaining correction terms of $G(a_i, p_i, \lambda_p)$, we impose that no more constraints arise from the contraction of the zero mode $\left(\tilde{\nu}^{(1)\tilde{\alpha}}_{(\nu)(a)}\right)$ with the gradient of potential $\tilde{V}^{(1)}(a_i, p_i, \lambda_p)$. This condition generates a general differential equation, which reads

$$\begin{aligned} 0 &= \tilde{\nu}^{(1)\tilde{\alpha}}_{(\nu)(a)} \frac{\partial \tilde{V}^{(1)}(a_i, p_i, \lambda_p)}{\partial \tilde{\xi}^{(1)\tilde{\alpha}}} \\ &= \mu^{\alpha}_{(\nu)} \left[\frac{\partial V^{(1)}(a_i, p_i)}{\partial \xi^{(1)\alpha}} + \frac{\partial G(a_i, p_i, \theta, \pi_\theta)}{\partial \xi^{(1)\alpha}} \right] + a \frac{\partial G(a_i, p_i, \lambda_p)}{\partial \theta} + b \frac{\partial G(a_i, p_i, \lambda_p)}{\partial \pi_\theta} \\ &= \mu^{\alpha}_{(\nu)} \left[\frac{\partial V^{(1)}(a_i, p_i)}{\partial \xi^{(1)\alpha}} + \sum_{m=0}^{\infty} \frac{\partial \mathcal{G}^{(m)}(a_i, p_i, \lambda_p)}{\partial \xi^{(1)\alpha}} \right] + a \sum_{n=0}^{\infty} \frac{\partial \mathcal{G}^{(n)}(a_i, p_i, \lambda_p)}{\partial \theta} \\ &\quad + b \sum_{m=0}^{\infty} \frac{\partial \mathcal{G}^{(n)}(a_i, p_i, \lambda_p)}{\partial \pi_\theta}. \end{aligned} \tag{33}$$

The last relation allows us to compute all correction terms in the order of λ_p, within $\mathcal{G}^{(n)}(a_i, p_i, \lambda_p)$. Note that this polynomial expansion in terms of λ_p is equal to zero; subsequently, all the coefficients for each order in this WZ variable must be identically null. In view of this, each correction term in the orders of λ_p can be determined as well. For a linear correction term, we have

$$0 = \mu^{\alpha}_{(\nu)} \left[\frac{\partial V^{(0)}(a_i, p_i)}{\partial \xi^{(1)\alpha}} + \frac{\partial \mathcal{G}^{(0)}(a_i, p_i)}{\partial \xi^{(1)\alpha}} \right] + a \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \lambda_p)}{\partial \theta} + b \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \lambda_p)}{\partial \pi_\theta}, \tag{34}$$

where the relation $V^{(1)} = V^{(0)}$ is used. For a quadratic correction term, we get

$$0 = \mu^{\alpha}_{(\nu)} \left[\frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \lambda_p)}{\partial \xi^{(0)\alpha}} \right] + a \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \lambda_p)}{\partial \theta} + b \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \lambda_p)}{\partial \pi_\theta}. \tag{35}$$

From these equations, a recursive equation for $n \geqslant 2$ is proposed as

$$0 = \mu^{\alpha}_{(\nu)} \left[\frac{\partial \mathcal{G}^{(n-1)}(a_i, p_i, \lambda_p)}{\partial \xi^{(0)\alpha}} \right] + a \frac{\partial \mathcal{G}^{(n)}(a_i, p_i, \lambda_p)}{\partial \theta} + b \frac{\partial \mathcal{G}^{(n)}(a_i, p_i, \lambda_p)}{\partial \pi_\theta}, \tag{36}$$

which allows us to compute the remaining correction terms in the order of θ and π_θ. This iterative process is successively repeated up to equation (33) when it becomes identically null (case (i)) or when an extra term $\mathcal{G}^{(n)}(a_i, p_i, \lambda_p)$ cannot be computed (case (ii)). Then, the new symplectic potential is written as

$$\tilde{V}^{(1)}(a_i, p_i, \lambda_p) = V^{(0)}(a_i, p_i) + G(a_i, p_i, \lambda_p). \tag{37}$$

For case (i), the new symplectic potential is gauge invariant. For the second case (ii), due to some correction terms within $G(a_i, p_i, \lambda_p)$ that are not yet determined, this new symplectic potential is not gauge invariant. As a consequence, there are some WZ counter-terms in the new symplectic potential, which can be fixed using Hamilton's equation of motion for the WZ variables θ and π_θ together with the canonical momentum relation conjugated to θ, given in equation (14). Due to this, the gauge-invariant Hamiltonian is obtained explicitly and the zero-mode $\tilde{\nu}^{(1)\tilde{\alpha}}_{(\nu)(a)}$ is identified as the generator of the infinitesimal gauge transformation given by

$$\delta \tilde{\xi}^{\tilde{\alpha}}_{(\nu)(a)} = \varepsilon \tilde{\nu}^{(1)\tilde{\alpha}}_{(\nu)(a)}, \tag{38}$$

where ε is an infinitesimal parameter.

3. Symplectic quantization of the noncommutative massive $U(1)$ theory

The Lagrangian density that governs the dynamics of the noncommutative massive $U(1)$ theory is

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} m^2 A^\mu A_\mu, \tag{39}$$

where the stress tensor in terms of the Moyal commutator is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}e[A_\mu, A_\nu], \tag{40}$$

with

$$[A_\mu, A_\nu] = A_\mu \star A_\nu - A_\nu \star A_\mu, \tag{41}$$

and where

$$\begin{aligned} A_\mu(x) \star A_\nu(x) &= \exp\left(\frac{\mathrm{i}}{2}\theta^{\gamma\lambda}\partial^x_\gamma \partial^y_\lambda\right) A_\mu(x) A_\nu(y)\big|_{x=y}, \\ A_\nu(x) \star A_\mu(x) &= \exp\left(\frac{\mathrm{i}}{2}\theta^{\lambda\gamma}\partial^x_\lambda \partial^y_\gamma\right) A_\nu(x) A_\mu(y)\big|_{x=y}, \end{aligned} \tag{42}$$

where $\theta^{\gamma\lambda}$ is a real and antisymmetric constant matrix. In order to avoid causality and unitary problems in the Moyal space, we take $\theta^{0i} = 0$ [17]. Hence, the $\star$ product of the gauge fields into the stress tensor, given in equations (40), becomes

$$\begin{aligned} A_\mu(x) \star A_\nu(x) &= \exp\left(\frac{\mathrm{i}}{2}\theta^{ij}\partial^x_i \partial^y_j\right) A_\mu(x) A_\nu(y)\big|_{x=y}, \\ A_\nu(x) \star A_\mu(x) &= \exp\left(\frac{\mathrm{i}}{2}\theta^{ji}\partial^x_j \partial^y_i\right) A_\nu(x) A_\mu(y)\big|_{x=y}. \end{aligned} \tag{43}$$

Now, we are ready to reduce the second-order Lagrangian density, equation (39), into its first-order form, which is read as

$$\begin{aligned} \mathcal{L} = {} & \pi^i \dot{A}_i + A_0(\partial_i \pi^i + m^2 A^0) + \tfrac{1}{2}\pi_i \pi^i - \mathrm{i}e\pi^i (A_0 \star A_i - A_i \star A_0) \\ & -\tfrac{1}{4} F_{ij} F^{ij} + \tfrac{1}{2} m^2 A_i A^i - \tfrac{1}{2} m^2 A_0 A^0, \end{aligned} \tag{44}$$

with the canonical momentum π_i given by

$$\begin{aligned}\pi_i &= -F_{0i}\\ &= -\dot{A}_i + \partial_i A_0 + \mathrm{i}e(A_0 \star A_i - A_i \star A_0).\end{aligned} \tag{45}$$

The symplectic fields are $\xi^{(0)\alpha} = (A^i, \pi^i, A^0)$ and the zeroth-iterative symplectic matrix is

$$f^{(0)} = \begin{pmatrix} 0 & -\delta^i_j & 0 \\ \delta^j_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(x-y), \tag{46}$$

which is a singular matrix. It has a zero mode that generates the following constraint:

$$\Omega(x) = \partial^x_i \pi^i(x) + m^2 A^0(x) - \mathrm{i}e[A_i(x), \pi^i(x)], \tag{47}$$

identified as the Gauss law. Bringing back this constraint into the canonical part of the first-order Lagrangian density $\mathcal{L}^{(0)}$ using a Lagrangian multiplier (β), the first-iterated Lagrangian density written in terms of $\xi^{(1)\alpha} = (A^i, \pi^i, A^0, \beta)$ is obtained as

$$\mathcal{L}^{(1)} = \pi^i \dot{A}_i + \dot{\beta}\Omega + \tfrac{1}{2}\pi_i\pi^i - \tfrac{1}{4}F_{ij}F^{ij} + \tfrac{1}{2}m^2 A_i A^i - \tfrac{1}{2}m^2 A_0 A^0, \tag{48}$$

with the following symplectic fields $\xi^{(1)\alpha} = (A^i, \pi^i, A^0, \beta)$. The first-iterated symplectic matrix is obtained as

$$f^{(1)} = \begin{pmatrix} 0 & -\delta^i_j\delta(x-y) & 0 & \mathrm{i}e[\pi^i(y), \delta(x-y)] \\ \delta^j_i\delta(x-y) & 0 & 0 & f_{\pi_i\beta} \\ 0 & 0 & 0 & m^2\delta(x-y) \\ \mathrm{i}e[\delta(x-y), \pi^i(x)] & f_{\beta\pi_i} & -m^2\delta(x-y) & 0 \end{pmatrix}, \tag{49}$$

where

$$f_{\pi_i\beta}(x,y) = \partial^y_i \delta(x-y) + \mathrm{i}e[\delta(x-y), A^i(y)]. \tag{50}$$

This matrix is nonsingular and, as set by the symplectic formalism, the Dirac brackets among the phase space fields are obtained from the inverse of the symplectic matrix, namely,

$$\begin{aligned}\{A_i(x), A^j(y)\}^* &= 0,\\ \{A_i(x), \pi^j(y)\}^* &= \delta^j_i\delta(x-y),\\ \{A_i(x), A_0(y)\}^* &= -\frac{1}{m^2}\left(\partial^x_i\delta(x-y) + \frac{\mathrm{i}e}{m^2}[\delta(x-y), A_i(y)]\right),\\ \{\pi^i(x), A_0(y)\}^* &= \frac{\mathrm{i}e}{m^2}[\delta(x-y), \pi^i(y)].\end{aligned} \tag{51}$$

Now, we are ready to implement the symplectic embedding formalism of the theory. This will be done in the next section.

4. The embedded model

At this point, we are interested to embed the massive noncommutative $U(1)$ theory via the symplectic embedding formalism (section 2) [1]. The symplectic embedding process begins enlarging the phase space with the introduction of two WZ fields $\gamma = (\eta, \pi_\eta)$. Due to this, the original Lagrangian density, equation (39), becomes

$$\tilde{\mathcal{L}} = \mathcal{L} + \mathcal{L}_{\mathrm{WZ}}, \tag{52}$$

where $\mathcal{L}_{\rm WZ}$ is a WZ counter-term which eliminates the noninvariance of the theory. In agreement with symplectic embedding formalism, this new Lagrangian density must be reduced into its first-order form, namely,

$$\tilde{\mathcal{L}}^{(0)} = \pi^i \dot{A}_i + \pi_\eta \dot{\eta} - \tilde{V}^{(0)}, \tag{53}$$

where $\tilde{V}^{(0)}$ is the symplectic potential given by

$$\begin{aligned}\tilde{V}^{(0)} &= -A_0(\partial_i \pi^i + m^2 A^0) - \tfrac{1}{2}\pi_i \pi^i + \mathrm{i}e\pi^i (A_0 \star A_i - A_i \star A_0) \\ &\quad + \tfrac{1}{4}F_{ij}F^{ij} - \tfrac{1}{2}m^2 A_i A^i + \tfrac{1}{2}m^2 A_0 A^0 + G(A^i, \pi^i, A^0, \gamma),\end{aligned} \tag{54}$$

where $G \equiv G(A^i, \pi^i, A^0, \gamma)$ is an arbitrary function and is written as an expansion in terms of the WZ fields as

$$G(A^i, \pi^i, A^0, \gamma) = \sum_{n=0}^{\infty} \mathcal{G}^{(n)}(A^i, \pi^i, A^0, \gamma) \qquad \text{with} \quad \mathcal{G}^{(n)}(A^i, \pi^i, A^0, \gamma) \sim (\gamma)^n. \tag{55}$$

The new symplectic variables are now given by $\tilde{\nu}^{(0)\alpha} = (A^i, \pi^i, A^0, \gamma)$ and the respective symplectic tensor is

$$\tilde{f}^{(0)} = \begin{pmatrix} 0 & -\delta^i_j \delta(x-y) & 0 & 0 & 0 \\ \delta^j_i \delta(x-y) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta(x-y) \\ 0 & 0 & 0 & \delta(x-y) & 0 \end{pmatrix}. \tag{56}$$

This singular matrix has a zero mode, which is set as

$$\tilde{\nu}^{(0)} = (0 \quad 0 \quad 1 \quad 0 \quad 0). \tag{57}$$

This zero mode when contracted with the symplectic potential generates the following constraint:

$$\Omega(x) = \partial^x_i \pi^i(x) + m^2 A^0(x) - \mathrm{i}e[A^i(x), \pi_i(x)] - \int \mathrm{d}y \frac{\delta G(y)}{\delta A^0(x)}. \tag{58}$$

In accordance with the symplectic scheme, this constraint must be introduced into the zeroth-iterative first-order Lagrangian density through a Lagrange multiplier ζ, generating the next one,

$$\tilde{\mathcal{L}}^{(1)} = \pi^i \dot{A}_i + \pi \dot{\eta} + \Omega \dot{\zeta} - \tilde{V}^{(1)}, \tag{59}$$

with $\tilde{V}^{(1)} = \tilde{V}^{(0)}|_{\Omega=0}$. Now, the symplectic vector is $\tilde{\xi}^{(1)} = (A^i, \pi^i, A^0, \zeta, \gamma)$ with the corresponding tensor given by

$$\tilde{f}^{(1)} = \begin{pmatrix} 0 & -\delta^j_i \delta(x-y) & 0 & \frac{\delta\Omega(y)}{\delta A^i(x)} & 0 & 0 \\ \delta^i_j \delta(x-y) & 0 & 0 & \frac{\delta\Omega(y)}{\delta \pi^i(x)} & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta\Omega(y)}{\delta A^0(x)} & 0 & 0 \\ -\frac{\delta\Omega(x)}{\delta A^j(y)} & -\frac{\delta\Omega(x)}{\delta \pi^j(y)} & -\frac{\delta\Omega(x)}{\delta A^0(y)} & 0 & -\frac{\delta\Omega(x)}{\delta \eta(y)} & -\frac{\delta\Omega(x)}{\delta \pi_\eta(y)} \\ 0 & 0 & 0 & \frac{\delta\Omega(y)}{\delta \eta(x)} & 0 & -\delta(x-y) \\ 0 & 0 & 0 & \frac{\delta\Omega(y)}{\delta \pi_\eta(x)} & \delta(x-y) & 0 \end{pmatrix}. \tag{60}$$

Now, we are ready to remove the noninvariance character of the original theory. To this end, it is necessary to assume that the symplectic matrix above is singular. Consequently, this matrix has a respective zero mode, which is degenerate due to the arbitrariness present in the matrix, which resides on the degenerate function G. This is not bad at all since it gives room for several approaches to eliminate the noninvariance structure. Consequently, this opens up the possibility of obtaining several commutative embedded representations for the noncommutative model, all of which are dynamically equivalent. This represents a quite simple feature of the symplectic embedding formalism. In view of this, we choose a convenient zero mode, which is written as

$$\tilde{\nu}^{(1)} = (\partial^{x,i} \quad 0 \quad 0 \quad 1 \quad 1 \quad 1). \tag{61}$$

Contracting this zero mode with the symplectic matrix above, namely,

$$\int \mathrm{d}^3x\, \nu^{(1)\tilde{\alpha}}(x) \tilde{f}^{(1)}_{\tilde{\alpha}\tilde{\beta}}(x, y) = 0, \tag{62}$$

we get the boundary condition $\mathcal{G}^{(0)}$ as

$$\mathcal{G}^{(0)}(x) = \tfrac{1}{2} m^2 A_0(x) A^0(x) - \mathrm{i}e[A^i(x), \pi_i(x)] A^0(x), \tag{63}$$

and some of the correction terms belong to $\mathcal{G}^{(1)}$, namely, $\pi_\eta A^0 - \eta A^0$. We note that the correction term $\mathcal{G}^{(n)}$ for $n \geqslant 2$ has no dependence on the temporal component of the potential field A^0. Thus, $\mathcal{G}^{(n)} \equiv \mathcal{G}^{(n)}(A^i, \pi^i, \gamma)$ for $n \geqslant 2$. This completes the first step of the symplectic embedding formalism.

After introducing these correction terms into the symplectic potential $\tilde{V}^{(1)}$, let us begin with the second step in order to reformulate the theory as a gauge theory. Following the prescription of the symplectic embedding formalism, the zero-mode $\tilde{\nu}^{(1)}$ does not produce a constraint when contracted with the gradient of symplectic potential, namely,

$$\int \mathrm{d}^3x\, \tilde{\nu}^{(1)\tilde{\alpha}}(y) \frac{\delta \tilde{V}^{(1)}(x)}{\delta \tilde{\xi}^{\tilde{\alpha}}(y)} = 0, \tag{64}$$

instead it produces a general equation that allows the computation of the correction terms in the order of γ enclosed into $G(A_i, \pi_i, A_0, \gamma)$. To compute the remaining linear correction terms expressed in the order of γ, $\mathcal{G}^{(1)}$, we use the following relation (see (34)):

$$0 = \int \mathrm{d}^3x \left\{ -\mathrm{i}e[F_{ij}(x), A^i(x)] \partial^j_y \delta(x - y) + m^2 A_j(x) \partial^j_y \delta(x - y) + \frac{\delta \mathcal{G}^{(1)}(x)}{\delta \eta(y)} + \frac{\delta \mathcal{G}^{(1)}(x)}{\delta \pi_\eta(y)} \right\}. \tag{65}$$

After a straightforward calculation, the complete linear correction term expressed in the order of γ is given by

$$\mathcal{G}^{(1)}(x) = \pi_\eta(x) A^0(x) - \eta(x) A^0(x) + \left\{ \mathrm{i}e \partial^j_x [F_{ij}(x), A^i(x)] - m^2 \partial^j_x A_j(x) \right\} \tfrac{1}{2} (\eta(x) + \pi_\eta(x)). \tag{66}$$

In order to compute the quadratic correction term, we use the following relation (see (35)):

$$\int \mathrm{d}^3x \left[\partial^j_x \left(\frac{\delta \mathcal{G}^{(1)}}{\delta A^j(x)} \right) + \frac{\delta \mathcal{G}^{(2)}(y)}{\delta \eta(x)} + \frac{\delta \mathcal{G}^{(2)}(y)}{\delta \pi_\eta(x)} \right] = 0, \tag{67}$$

which after a direct calculation gives

$$\begin{aligned}
\mathcal{G}^{(2)}(x) = &-\frac{\mathrm{i}e}{4} F_{ij} \big[\partial^i_x \eta(x), \partial^j_x \eta(x) \big] - \frac{1}{4} m^2 \partial^j_x \eta(x) \partial^x_j \eta(x) \\
&- \frac{e^2}{4} \big[\partial^i_x \eta(x), A_j(x) \big] \big[A_i(x), \partial^j_x \eta(x) \big] - \frac{e^2}{4} \big[A_i(x), \partial^x_j \eta(x) \big] \big[A^i(x), \partial^j_x \eta(x) \big] \\
&- \frac{\mathrm{i}e}{4} F_{ij} \big[\partial^i_x \pi_\eta(x), \partial^j_x \pi_\eta(x) \big] - \frac{1}{4} m^2 \partial^j_x \pi_\eta(x) \partial^x_j \pi_\eta(x)
\end{aligned}$$

$$
- \frac{e^2}{4}\big[\partial_x^i \pi_\eta(x), A_j(x)\big]\big[A_i(x), \partial_x^j \pi_\eta(x)\big]
- \frac{e^2}{4}\big[A_i(x), \partial_j^x \pi_\eta(x)\big]\big[A^i(x), \partial_x^j \pi_\eta(x)\big]. \tag{68}
$$

The remaining correction terms $\mathcal{G}^{(n)}$ for $n \geqslant 3$ are computed in an analogous way (see (36)) and we just write them down as

$$
\mathcal{G}^{(3)}(x) = \frac{e^2}{2}\big[A_i(x), \partial_j^x \eta(x)\big]\big[\partial_x^j \eta(x), \partial^{y,i}\eta(x)\big] + \frac{e^2}{2}\big[A_i(x), \partial_j^x \pi_\eta(x)\big]\big[\partial_x^j \pi_\eta(x), \partial^{y,i}\pi_\eta(x)\big],
$$
$$
\mathcal{G}^{(4)}(x) = \frac{e^2}{8}\big[\partial_i^x \eta(x), \partial_j^x \eta(x)\big]\big[\partial_x^j \eta(x), \partial_x^i \eta(x)\big] + \frac{e^2}{8}\big[\partial_i^x \pi_\eta(x), \partial_j^x \pi_\eta(x)\big]\big[\partial_x^j \pi_\eta(x), \partial_x^i \pi_\eta(x)\big]. \tag{69}
$$

Note that the fourth-order correction term has dependence on the WZ field only, thus all correction terms $\mathcal{G}^{(n)}$ for $n \geqslant 5$ are null. Then, the gauge-invariant first-order Lagrangian density is given by

$$
\tilde{\mathcal{L}} = \pi^i(x)\dot{A}_i(x) + \pi(x)\dot{\eta}(x) - \tilde{\mathcal{H}}, \tag{70}
$$

where $\tilde{\mathcal{H}}$ is the gauge-invariant Hamiltonian density, identified as the symplectic potential $\tilde{V}^{(1)}$, namely,

$$
\begin{aligned}
\tilde{\mathcal{H}} = \tilde{V}^{(1)} = & -\frac{1}{2}\pi_i(x)\pi^i(x) + \frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}m^2 A_i(x)A^i(x) + \pi_\eta(x)A^0(x) - \eta(x)A^0(x) \\
& + \frac{\mathrm{i}e}{2}\partial_x^j[F_{ij}(x), A^i(x)]\eta(x) + \frac{1}{2}m^2\partial_x^j A_j(x)\eta(x) - \frac{\mathrm{i}e}{4}F_{ij}\big[\partial_x^i \eta(x), \partial_x^j \eta(x)\big] \\
& - \frac{1}{4}m^2 \partial_x^j \eta(x)\partial_j^x \eta(x) - \frac{e^2}{4}\big[\partial_x^i \eta(x), A_j(x)\big]\big[A_i(x), \partial_x^j \eta(x)\big] \\
& - \frac{e^2}{4}\big[A_i(x), \partial_j^x \eta(x)\big]\big[A^i(x), \partial_x^j \eta(x)\big] + \frac{e^2}{2}\big[A_i(x), \partial_j^x \eta(x)\big]\big[\partial_x^j \eta(x), \partial^{y,i}\eta(x)\big] \\
& + \frac{e^2}{8}\big[\partial_i^x \eta(x), \partial_j^x \eta(x)\big]\big[\partial_x^j \eta(x), \partial_x^i \eta(x)\big] + \frac{\mathrm{i}e}{2}\partial_x^j[F_{ij}(x), A^i(x)]\pi_\eta(x) \\
& + \frac{1}{2}m^2\partial_x^j A_j(x)\pi_\eta(x) - \frac{\mathrm{i}e}{4}F_{ij}\big[\partial_x^i \pi_\eta(x), \partial_x^j \pi_\eta(x)\big] - \frac{1}{4}m^2\partial_x^j \pi_\eta(x)\partial_j^x \pi_\eta(x) \\
& - \frac{e^2}{4}\big[\partial_x^i \pi_\eta(x), A_j(x)\big]\big[A_i(x), \partial_x^j \pi_\eta(x)\big] \\
& - \frac{e^2}{4}\big[A_i(x), \partial_j^x \pi_\eta(x)\big]\big[A^i(x), \partial_x^j \pi_\eta(x)\big] \\
& + \frac{e^2}{2}\big[A_i(x), \partial_j^x \pi_\eta(x)\big]\big[\partial_x^j \pi_\eta(x), \partial^{y,i}\pi_\eta(x)\big] \\
& + \frac{e^2}{8}\big[\partial_i^x \pi_\eta(x), \partial_j^x \pi_\eta(x)\big]\big[\partial_x^j \pi_\eta(x), \partial_x^i \pi_\eta(x)\big].
\end{aligned} \tag{71}
$$

In order to complete the gauge-invariant reformulation of the massive noncommutative $U(1)$ theory, we compute the infinitesimal gauge transformations of the phase space coordinates. In agreement with the symplectic method, the zero-mode $\tilde{\nu}^{(1)}$ is the generator of the infinitesimal gauge transformations ($\delta\mathcal{O} = \varepsilon\tilde{\nu}^{(1)}$), which are given by

$$
\delta A_i = \partial^i \varepsilon, \qquad \delta\pi^i = 0, \qquad \delta A_0 = 0, \qquad \delta\eta = \varepsilon, \qquad \delta\pi_\eta = \varepsilon, \tag{72}
$$

where $\varepsilon(y)$ is an infinitesimal time-dependent parameter. Thus, we complete the Hamiltonian symplectic embedding of the massive noncommutative $U(1)$ theory.

Despite the result, it seems important to mention here that it is well known that the gauge invariance can also be obtained by means of Stückelberg fields [18]. Indeed, some authors [2] have discussed this point, precisely, in section 2 of [2]. These authors have rewritten the potential fields as

$$\bar{A}_\mu = g \star A_\mu \star g^{-1} + \imath g \star \partial_\mu \star g^{-1}, \tag{73}$$

where g is the Stückelberg field. Subsequently, the Lagrangian, equation (39), rewritten in terms of the modified gauge fields $\bar{A}_\mu$ becomes a gauge-invariant description of the massive noncommutative $U(1)$ theory, whose infinitesimal gauge transformations are

$$\begin{aligned} \delta A_\mu &= \partial_\mu \alpha - \mathrm{i}e[A_\mu, \alpha], \\ \delta g &= -\mathrm{i}eg \star \alpha. \end{aligned} \tag{74}$$

In [2], the authors also mentioned that it is not trivial to relate the gauge-invariant Lagrangian, obtained via the Stückelberg formalism, to the one obtained through the BFFT constrained conversion scheme. In other work [19], the authors demonstrated that the gauge-invariant Lagrangian for the massive commutative $U(1)$ theory obtained through BFFT and the Stückelberg formalism are equivalent in a quite easy form. Due to this, it seems that the requirement of an infinite number of steps demanded by the BFFT method to change the second class nature of the constraints to first class leads to a nontrivial connection with the result obtained when the Stückelberg formalism is used.

In the present work, the second class nature of the constraints was changed to first class by means of a finite number of steps and the infinitesimal gauge transformations, pointed out by the zero mode, are picked out in order to make the constraint conversion process easy. Due to this, we argue that if we choose a specific infinitesimal gauge transformation (zero mode), it is possible to reproduce the result obtained by the Stückelberg formalism. Hence, it seems possible to make trivial the connection of the gauge-invariant Lagrangian description, obtained by the Stückelberg formalism, with the one obtained by the introduction of WZ fields.

5. Conclusion

In this paper, we have embedded the massive noncommutative $U(1)$ theory. This was achieved by an alternative embedding formalism based on the symplectic framework. The Hamiltonian density of the embedded version of the massive noncommutative $U(1)$ theory was also obtained. A remarkable feature mentioned in the present work is that the embedded version was obtained by applying a finite number of steps in the iterative symplectic embedding process, which leads to an embedded Hamiltonian density with a finite number of WZ terms. Further, we also discuss, in section 3, the new possibilities of embedding the noncommutative theory. It is important to note that, by construction, these different embedded representations of the noncommutative theory are dynamically equivalent, since the WZ gauge orbit is defined by the zero mode. In view of this, the symplectic embedding formalism seems powerful enough when compared with other WZ conversion schemes.

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Appendix. Some properties of the Moyal product

In this appendix we list some properties that we use in this paper.

$$\int \mathrm{d}^n x\, \phi_1 \star \phi_2 = \int \mathrm{d}^n x\, \phi_1 \phi_2 = \int \mathrm{d}^n x\, \phi_2 \star \phi_1, \tag{A.1}$$

$$(\phi_1 \star \phi_2) \star \phi_3 = \phi_1 \star (\phi_2 \star \phi_3) = \phi_1 \star \phi_2 \star \phi_3, \tag{A.2}$$

$$\begin{aligned}\int \mathrm{d}^n x\, \phi_1 \star \phi_2 \star \phi_3 &= \int \mathrm{d}^n x\, \phi_2 \star \phi_3 \star \phi_1 \\ &= \int \mathrm{d}^n x\, \phi_3 \star \phi_1 \star \phi_2,\end{aligned} \tag{A.3}$$

$$\begin{aligned}\int \mathrm{d}^n x\, [[A, B], C] \star D &= \int \mathrm{d}^n x\, [A, B] \star [C, D] \\ &= \int \mathrm{d}^n x\, [A, B][C, D].\end{aligned} \tag{A.4}$$

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